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# General supersymmetric $AdS_5$ black holes

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Abstract: Supersymmetric, asymptotically $AdS_5$, black hole solutions of five dimensional gauged supergravity coupled to arbitrarily many abelian vector multiplets are presented. The general nature of supersymmetric solutions of this theory is discussed. All maximally supersymmetric solutions of this theory (with or without gauging) are obtained.

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## 1. Introduction

An obvious requirement of any theory of quantum gravity is to provide a microscopic explanation of black hole entropy. To some extent, the AdS/CFT correspondence [1] has done this: the microstates of an asymptotically anti-de Sitter (AdS) black hole are states of a dual CFT. In three dimensions, it is possible to calculate the entropy of black holes by counting CFT states with the same quantum numbers as the black hole using the Cardy formula [2]. However, an analagous calculation in more than three dimensions has been hindered by our poor understanding of strongly coupled quantum field theory.

A potential way of overcoming this obstacle is to consider asymptotically AdS black holes preserving enough supersymmetry that the corresponding CFT states belong to short superconformal multiplets. The number of such states with given quantum numbers is not expected to change as the CFT coupling is varied, so by counting them at weak coupling it should be possible to reproduce the black hole entropy.

Four dimensional, supersymmetric, asymptotically AdS, black holes were presented in [3]. However, a CFT calculation of their entropy does not seem possible because so little is known about the dual CFT in this case. Seven dimensional supersymmetric AdS black holes were constructed in [4]. Once again, our ignorance of the dual CFT gives us little hope of being able to calculate their entropy.

This leads us to $D=5$, where our understanding of the CFT, $\mathcal{N}=4$ SU($N$) super Yang-Mills theory, is much better. The first examples of supersymmetric, asymptotically $AdS_5$, black holes were recently obtained in [5] as solutions of minimal $D=5$ gauged supergravity. These solutions have to rotate, just like supersymmetric AdS black holes

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in $D = 3, 4$ (but unlike $D = 7$). They preserve at least two supersymmetries, and their mass $M$ and angular momenta $J_{1,2}$ are functions of their electric charge $Q$: $M = M(Q)$, $J_1 = J_2 = J(Q)$. These black holes should correspond to SU($N$) singlet CFT states on $R \times S^3$. The CFT state/operator correspondence maps such states to gauge-invariant local operators at the origin of (euclidean) $R^4$ [6]. In order to identify the local operators corresponding to the black hole, the solution must first be oxidized to give an asymptotically $AdS_5 \times S^5$ solution of type-IIB supergravity. This can be done using results of [7]. One finds that the dual CFT operators are 1/16 BPS, have $\mathrm{SO}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ spins $J_L = J(Q)$, $J_R = 0$ and R-charge given by the SU(4) weight vector $(Q, Q, Q)$ (for suitably normalized $Q$). We expect that it should be possible to reproduce the black hole entropy by counting such operators at weak coupling.

It is natural to expect that there should be more general supersymmetric black hole solutions for which the corresponding CFT operators have R-charge $(Q_1, Q_2, Q_3)$. To see how these should arise, note that type-IIB supergravity can be dimensionally reduced on $S^5$ to give $\mathcal{N} = 1$ $D = 5$ gauged supergravity coupled to three abelian vector multiplets, with gauge group $\mathrm{U}(1)^3$ [8].[1] This theory has eight supercharges. Black holes carrying U(1) charges $Q_i$ will correspond to CFT operators with R-charge $(Q_1, Q_2, Q_3)$. The goal of the present paper is to find supersymmetric, asymptotically $AdS_5$, black hole solutions with these charges.

Although we are mainly interested in the $\mathrm{U}(1)^3$ theory, we shall work within the more general framework of $\mathcal{N} = 1$ $D = 5$ gauged supergravity coupled to arbitrarily many abelian vector multiplets [9]. One possible way to find black hole solutions would be to use the results of [5] to motivate an Ansatz for the fields of this theory, and then examine the circumstances under which this Ansatz admits a super-covariantly constant spinor. However, we shall adopt a more systematic approach in which we analyze the general nature of supersymmetric solutions of this theory. This approach was first used in [10] for minimal $\mathcal{N} = 2$, $D = 4$ supergravity and has been applied recently to minimal $D = 5$ [11] and $D = 6$ [12] supergravity, minimal $D = 5$ gauged supergravity [13], and minimal $\mathcal{N} = 2$, $D = 4$ gauged supergravity [14]. It has also been used in $D = 11$ supergravity [15, 16]. In this way, not only do we find the black hole solutions, we also obtain a general framework which may be used to construct interesting new solutions, many of which cannot be found easily by guessing Ansätze.

In section 2 of this paper, we generalize some of the results of [11, 13] to include arbitrarily many abelian vector multiplets. Our first result is a complete classification of all maximally supersymmetric solutions in both the gauged and ungauged theory. In the gauged theory, the only maximally supersymmetric solution is $AdS_5$ with vanishing gauge fields and prescribed values for the scalars. In the ungauged theory, we find that the scalars take arbitrary constant values but once these values have been chosen, the maximally supersymmetric solutions are in one-to-one correspondence with the maximally supersymmetric solutions of the minimal theory, which were given in [11].

[1] Strictly speaking, there are only two vector multiplets since the diagonal U(1) arises from the graviphoton.



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We then turn our attention to general supersymmetric solutions. Just as in the minimal theories [11, 13], such solutions possess a globally defined Killing vector field that is either everywhere null, or timelike somewhere. We consider only solutions in the latter class, and show that they admit a hyper-Kähler or Kähler structure in the ungauged and gauged theories respectively, again just as in the minimal theories [11, 13]. We determine a general form for such solutions of the gauged theory and show that they must preserve at least two supersymmetries.

In section 3, we use these results to derive supersymmetric black hole solutions of $\mathcal{N}=1$, $D=5$ gauged supergravity coupled to arbitrarily many abelian vector multiplets. Just like the solutions of [5], these are asymptotically AdS solutions that are parametrized by their electric charges, and have two equal angular momenta. We pay special attention to solutions of the $\mathrm{U}(1)^3$ theory mentioned above since these can be oxidized to give BPS, asymptotically $AdS_5 \times S^5$ solutions of IIB supergravity.

We should note that supersymmetric solutions of $D=5$ gauged supergravity coupled to vector multiplets have been discussed before. In particular, there have been attempts at finding supersymmetric black hole solutions of this theory, but these attempts only produced solutions with naked singularities [17] or naked closed timelike curves [18]. Some black string solutions were presented in [19].

The reader interested only in the black hole solutions (and not their derivation) should jump to subsection 3.2.

## 2. Supersymmetric solutions of $\mathcal{N}=1$ supergravity

### 2.1 $\mathcal{N}=1$ supergravity

The action of $\mathcal{N}=1$ $D=5$ gauged supergravity coupled to $n$ abelian vector multiplets with scalars taking values in a symmetric space is [9][2]

$$S = \frac{1}{16\pi G}\int \left( -{}^5R + 2\chi^2\mathcal{V} - Q_{IJ}F^I \wedge *F^J + Q_{IJ}dX^I \wedge \star dX^J - \frac{1}{6}C_{IJK}F^I \wedge F^J \wedge A^K \right) \tag{2.1}$$

where $I, J, K$ take values $1, \ldots, n$ and $F^I = dA^I$. $C_{IJK}$ are constants that are symmetric on $IJK$ and obey

$$C_{IJK}C_{J'(LM}C_{PQ)K'}\delta^{JJ'}\delta^{KK'} = \frac{4}{3}\delta_{I(L}C_{MPQ)}\,. \tag{2.2}$$

See [19] for a more detailed recent discussion of this theory. The $X^I$ are scalars which are constrained via

$$\frac{1}{6}C_{IJK}X^I X^J X^K = 1\,. \tag{2.3}$$

We may regard the $X^I$ as being functions of $n-1$ unconstrained scalars $\phi^a$. It is convenient to define

$$X_I \equiv \frac{1}{6}C_{IJK}X^J X^K \tag{2.4}$$

[2] We use a negative signature metric.



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so that the condition (2.3) becomes

$$X_I X^I = 1\,. \tag{2.5}$$

In addition, the coupling $Q_{IJ}$ depends on the scalars via

$$Q_{IJ} = \frac{9}{2} X_I X_J - \frac{1}{2} C_{IJK} X^K \tag{2.6}$$

so in particular

$$Q_{IJ} X^J = \frac{3}{2} X_I\,, \qquad Q_{IJ} \partial_a X^J = -\frac{3}{2} \partial_a X_I\,. \tag{2.7}$$

The constraints (2.2) are sufficient to ensure that the matrix $Q_{IJ}$ is invertible with inverse $Q^{IJ}$ given by

$$Q^{IJ} = 2X^I X^J - 6C^{IJK} X_K \tag{2.8}$$

where

$$C^{IJK} \equiv \delta^{II'} \delta^{JJ'} \delta^{KK'} C_{I'J'K'}\,. \tag{2.9}$$

It is then straightforward to show that

$$X^I = \frac{9}{2} C^{IJK} X_J X_K\,. \tag{2.10}$$

The scalar potential can be written as

$$\mathcal{V} = 27 C^{IJK} V_I V_J X_K \tag{2.11}$$

where $V_I$ are constants.

For a bosonic background to be supersymmetric there must be a spinor[3] $\epsilon^a$ for which the supersymmetry variations of the gravitino and dilatino vanish. For the gravitino this requires

$$\left[\nabla_\mu + \frac{1}{8} X_I (\gamma_\mu{}^{\nu\rho} - 4\delta_\mu{}^\nu \gamma^\rho) F^I{}_{\nu\rho}\right] \epsilon^a - \frac{\chi}{2} V_I (X^I \gamma_\mu - 3A^I{}_\mu) \epsilon^{ab} \epsilon^b = 0 \tag{2.12}$$

and for the dilatino it requires

$$\left[\left(\frac{1}{4} Q_{IJ} \gamma^{\mu\nu} F^J{}_{\mu\nu} + \frac{3}{4} \gamma^\mu \nabla_\mu X_I\right) \epsilon^a - \frac{3\chi}{2} V_I \epsilon^{ab} \epsilon^b\right] \frac{\partial X^I}{\partial \phi^a} = 0\,. \tag{2.13}$$

The Einstein equation is

$${}^5R_{\alpha\beta} + Q_{IJ} F^I{}_{\alpha\lambda} F^J{}_\beta{}^\lambda - Q_{IJ} \nabla_\alpha X^I \nabla_\beta X^J - \frac{1}{6} g_{\alpha\beta} \left(4\chi^2 \mathcal{V} + Q_{IJ} F^I{}_{\mu\nu} F^{J\mu\nu}\right) = 0\,. \tag{2.14}$$

The Maxwell equations (varying $A^I$) are

$$d\left(Q_{IJ} \star F^J\right) = -\frac{1}{4} C_{IJK} F^J \wedge F^K\,. \tag{2.15}$$

[3] We use symplectic Majorana spinors. Our conventions are the same as [11].



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The scalar equations (varying $\phi^a$) are

$$\Big[-d(\star dX_I) + \Big(X_M X^P C_{NPI} - \frac{1}{6}C_{MNI}\Big)(F^M \wedge \star F^N - dX^M \wedge \star dX^N) - \\ -6\chi^2 C^{MPQ} V_M V_P C_{QIJ} X^J \mathrm{dvol}\Big] \frac{\partial X^I}{\partial \phi^a} = 0\,. \tag{2.16}$$

If a quantity $L_I$ satisfies $L_I \partial_a X^I = 0$ then there must be a function $M$ such that $L_I = M X_I$. This implies that the dilatino equation (2.13) can be simplified to

$$\left[\left(\frac{1}{4}Q_{IJ} - \frac{3}{8}X_I X_J\right) F^J{}_{\mu\nu}\gamma^{\mu\nu} + \frac{3}{4}\gamma^\mu \nabla_\mu X_I\right]\epsilon^a + \frac{3\chi}{2}\left(X_I V_J X^J - V_I\right)\epsilon^{ab}\epsilon^b = 0\,, \tag{2.17}$$

and the scalar equation can be written as

$$\begin{aligned} -d\left(\star dX_I\right) + \left(\frac{1}{6}C_{MNI} - \frac{1}{2}X_I C_{MNJ}X^J\right) dX^M \wedge \star dX^N + \\ + \left(X_M X^P C_{NPI} - \frac{1}{6}C_{MNI} - 6X_I X_M X_N + \frac{1}{6}X_I C_{MNJ}X^J\right) F^M \wedge \star F^N + \\ +6\chi^2\left(-C^{MPQ}V_M V_P C_{QIJ}X^J + 6X_I C^{MPQ}V_M V_P X_Q\right)\mathrm{dvol} = 0\,. \end{aligned} \tag{2.18}$$

### 2.2 Maximal supersymmetry

All maximally supersymmetric solutions of the minimal ungauged and gauged $D = 5$ supergravity theories were explicitly obtained in [11] and [13] respectively. In the gauged theory, the unique maximally supersymmetric solution is $AdS_5$ with vanishing gauge field. The ungauged theory has a more complicated set of maximally supersymmetric solutions.[4]

We shall now identify all maximally supersymmetric solutions of $\mathcal{N} = 1$ $D = 5$ supergravity coupled to $n$ vector multiplets. First we examine the dilatino equation (2.17). For maximal supersymmetry, this equation must impose no algebraic constraints on the Killing spinor, which implies that the terms with two, one and zero gamma matrices must vanish independently. This gives

$$dX_I = 0 \tag{2.19}$$

so the scalars $X_I$ (and hence also $X^I$) are constant:

$$X_I \equiv \bar{X}_I\,. \tag{2.20}$$

Moreover,

$$\left(\frac{1}{4}Q_{IJ} - \frac{3}{8}\bar{X}_I\bar{X}_J\right) F^J = 0\,, \tag{2.21}$$

which implies that (at least locally)

$$A^I = \bar{X}^I A \tag{2.22}$$

[4] It was conjectured in [11] that some of the maximally supersymmetric solutions obtained there are isometric. This conjecture was proved in [21].



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for some 1-form $A$. Lastly, in the gauged theory ($\chi \neq 0$) we also obtain

$$\bar{X}_I = \xi^{-1} V_I \,, \tag{2.23}$$

where

$$\xi^3 = \frac{9}{2} C^{IJK} V_I V_J V_K \,. \tag{2.24}$$

In the ungauged theory, the values of $\bar{X}_I$ are arbitrary. It is convenient to define

$$F = dA \,. \tag{2.25}$$

Upon substituting these identities back into the equations of motion and gravitino equation, we observe that the scalar equation holds automatically. The other equations simplify to

$$^5R_{\alpha\beta} + \frac{3}{2} F_{\alpha\lambda} F_\beta{}^\lambda - \frac{1}{2} g_{\alpha\beta} \left( 8\chi^2 \xi^2 + \frac{1}{2} F_{\lambda\mu} F^{\lambda\mu} \right) = 0 \,, \tag{2.26}$$

$$d \star F + F \wedge F = 0 \,, \tag{2.27}$$

$$\left( \nabla_\mu + \frac{1}{8} (\gamma_\mu{}^{\nu\rho} - 4\delta_\mu{}^\nu \gamma^\rho) F_{\nu\rho} \right) \epsilon^a - \frac{\chi\xi}{2} (\gamma_\mu - 3A_\mu) \epsilon^{ab} \epsilon^b = 0 \,. \tag{2.28}$$

However, these are merely the equations of motion and gravitino equation of the *minimal* supergravity.[5] Hence the maximally supersymmetric solutions of the non-minimal theory are in one-to-one correspondence with those of the minimal theory.[6] In particular, in the gauged theory, the only maximally supersymmetric solution is $AdS_5$ with radius $\ell$ given by

$$\frac{1}{\ell} = \chi\xi \,. \tag{2.29}$$

### 2.3 General supersymmetric solutions

Following [11], our strategy for determining the general nature of bosonic supersymmetric solutions is to analyse the differential forms that can be constructed from a (commuting) Killing spinor. We first investigate algebraic properties of these forms, and then their differential properties.

From a single commuting spinor $\epsilon^a$ we can construct a scalar $f$, a 1-form $V$ and three 2-forms $\Phi^{ab} \equiv \Phi^{(ab)}$:

$$f \epsilon^{ab} = \bar{\epsilon}^a \epsilon^b \,, \qquad V_\alpha \epsilon^{ab} = \bar{\epsilon}^a \gamma_\alpha \epsilon^b \,, \qquad \Phi^{ab}_{\alpha\beta} = \bar{\epsilon}^a \gamma_{\alpha\beta} \epsilon^b \,. \tag{2.30}$$

$f$ and $V$ are real, but $\Phi^{11}$ and $\Phi^{22}$ are complex conjugate and $\Phi^{12}$ is imaginary. It is convenient to work with three real two-forms $J^{(i)}$ defined by

$$\Phi^{(11)} = J^{(1)} + iJ^{(2)} \,, \qquad \Phi^{(22)} = J^{(1)} - iJ^{(2)} \,, \qquad \Phi^{(12)} = -iJ^{(3)} \,. \tag{2.31}$$

[5] To compare with the conventions of [11, 13] one must make the replacements $A \to (2/\sqrt{3})A$ and $\chi\xi \to \chi/(2\sqrt{3})$.

[6] Strictly speaking, in the ungauged theory the solutions are also parametrized by the values of the constants $\bar{X}_I$.



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It will be useful to record some of the algebraic identities that can be obtained from the Fierz identity [11]:

$$V_\alpha V^\alpha = f^2\,, \tag{2.32}$$
$$J^{(i)} \wedge J^{(j)} = -2\delta_{ij} f \star V\,, \tag{2.33}$$
$$i_V J^{(i)} = 0\,, \tag{2.34}$$
$$i_V \star J^{(i)} = -f J^{(i)}\,, \tag{2.35}$$
$$J^{(i)}_{\gamma\alpha} J^{(j)\gamma}{}_\beta = \delta_{ij}\left(f^2\eta_{\alpha\beta} - V_\alpha V_\beta\right) + \epsilon_{ijk} f J^{(k)}_{\alpha\beta} \tag{2.36}$$

where $\epsilon_{123} = +1$ and, for a vector $Y$ and $p$-form $A$, $(i_Y A)_{\alpha_1,\ldots,\alpha_{p-1}} \equiv Y^\beta A_{\beta\alpha_1,\ldots,\alpha_{p-1}}$. Finally, we have

$$V_\alpha \gamma^\alpha \epsilon^a = f\epsilon^a \tag{2.37}$$
$$\Phi^{ab}_{\alpha\beta}\gamma^{\alpha\beta}\epsilon^c = 8f\epsilon^{c(a}\epsilon^{b)}\,. \tag{2.38}$$

Equation (2.32) implies that $V$ is timelike, null or zero. The final possibility can be eliminated using arguments in [11, 20].

We now turn to the differential conditions that arise because $\epsilon$ is a Killing spinor. We differentiate $f$, $V$, $\Phi$ in turn and use (2.12). Starting with $f$ we find

$$df = -i_V\left(X_I F^I\right), \tag{2.39}$$

which implies $\mathcal{L}_V f = 0$ where $\mathcal{L}$ denotes the Lie derivative. Next, differentiating $V$ gives

$$D_{(\alpha} V_{\beta)} = 0\,, \tag{2.40}$$

so $V$ is a Killing vector, and

$$dV = 2fX_I F^I + X_I \star (F^I \wedge V) + 2\chi V_I X^I J^{(1)}\,. \tag{2.41}$$

Finally, differentiating $J^{(i)}$ gives

$$\begin{aligned} D_\alpha J^{(i)}_{\beta\gamma} = & -\frac{1}{2}X_I\left[2F^I{}_\alpha{}^\delta\left(\star J^{(i)}\right)_{\delta\beta\gamma} - 2F^I{}_{[\beta}{}^\delta\left(\star J^{(i)}\right)_{\gamma]\alpha\delta} + \eta_{\alpha[\beta}F^{I\delta\epsilon}\left(\star J^{(i)}\right)_{\gamma]\delta\epsilon}\right] - \\ & -2\chi V_I X^I \delta^{i1}\eta_{\alpha[\beta}V_{\gamma]} + 3\chi\epsilon^{1ij}V_I\left[A^I{}_\alpha J^{(j)}{}_{\beta\gamma} + \frac{1}{3}X^I(\star J^{(j)})_{\alpha\beta\gamma}\right], \end{aligned} \tag{2.42}$$

which implies

$$dJ^{(i)} = 3\chi\epsilon^{1ij}V_I\left(A^I \wedge J^{(j)} + X^I \star J^{(j)}\right) \tag{2.43}$$

so $dJ^{(1)} = 0$ but $J^{(2)}$ and $J^{(3)}$ are only closed in the ungauged theory (i.e. when $\chi = 0$). Equation (2.43) implies

$$\mathcal{L}_V J^{(i)} = 3\chi\epsilon^{1ij}\left(i_V(V_I A^I) - V_I X^I f\right) J^{(j)}\,. \tag{2.44}$$



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Now consider the effect of a gauge transformation $A^I \to A^I + d\Lambda^I$. The Killing spinor equation is invariant provided the spinor transforms according to

$$\begin{aligned}\epsilon^1 &\to \cos\left(\frac{3\chi V_I \Lambda^I}{2}\right)\epsilon^1 - \sin\left(\frac{3\chi V_I \Lambda^I}{2}\right)\epsilon^2 \\ \epsilon^2 &\to \cos\left(\frac{3\chi V_I \Lambda^I}{2}\right)\epsilon^2 + \sin\left(\frac{3\chi V_I \Lambda^I}{2}\right)\epsilon^1 .\end{aligned} \tag{2.45}$$

Under these transformations, $f \to f$, $V \to V$ and $J^{(1)} \to J^{(1)}$, but $J^{(2)} + iJ^{(3)} \to e^{-3i\chi V_I \Lambda^I}(J^{(2)} + iJ^{(3)})$, so $J^{(2,3)}$ are only gauge-invariant in the ungauged theory. We shall choose to work in a gauge in which

$$i_V A^I = f X^I . \tag{2.46}$$

In such a gauge we have $\mathcal{L}_V J^{(i)} = 0$.

To make further progress we will examine the dilatino equation (2.17). Contracting with $\bar{\epsilon}^c$ we obtain

$$\mathcal{L}_V X_I = 0 \tag{2.47}$$

and

$$\left(\frac{1}{4}Q_{IJ} - \frac{3}{8}X_I X_J\right) F^J{}_{\mu\nu}(J^{(i)})^{\mu\nu} = -\frac{3\chi}{2}\delta^{1i}(X_I V_J X^J - V_I) f . \tag{2.48}$$

Next, contracting (2.17) with $\bar{\epsilon}^c \gamma^\sigma$ we find

$$i_V F^J = -d(f X^J) , \tag{2.49}$$

which implies that

$$\mathcal{L}_V F^J = 0 . \tag{2.50}$$

Hence $V$ generates a symmetry of all of the fields. In the gauge (2.46) we also have

$$\mathcal{L}_V A^I = 0 . \tag{2.51}$$

Contracting (2.17) with $\bar{\epsilon}^c \gamma^\sigma$ we obtain the identity

$$\left(\frac{1}{4}Q_{IJ} - \frac{3}{8}X_I X_J\right) F^J{}_{\mu\nu}(\star J^{(i)})_\sigma{}^{\mu\nu} = -\frac{3}{4}(J^{(i)})_\sigma{}^\mu \nabla_\mu X_I - \frac{3\chi}{2}\delta^{i1}(X_I V_J X^J - V_I) V_\sigma . \tag{2.52}$$

Finally, contracting (2.17) with $\bar{\epsilon}^c \gamma^{\sigma\lambda}$ gives

$$\begin{aligned}\left(\frac{1}{4}Q_{IJ} - \frac{3}{8}X_I X_J\right)\left(F^J{}_{\mu\nu}(\star V)^{\sigma\lambda\mu\nu} + 2F^{J\lambda\sigma}\right) = &-\frac{3}{4}(\nabla^\lambda X_I V^\sigma - \nabla^\sigma X_I V^\lambda) + \\ &+\frac{3\chi}{2}(X_I V_J X^J - V_I)(J^{(1)})^{\sigma\lambda}\end{aligned} \tag{2.53}$$

and

$$\begin{aligned}\left(\frac{1}{2}Q_{IJ} - \frac{3}{4}X_I X_J\right)\left(F^{J\sigma}{}_\nu (J^{(i)})^{\nu\lambda} - F^{J\lambda}{}_\nu (J^{(i)})^{\nu\sigma}\right) = &-\frac{3}{4}\nabla_\mu X_I (\star J^{(i)})^{\sigma\lambda\mu} + \\ &+\frac{3\chi}{2}\epsilon^{1ij}(X_I V_J X^J - V_I)\left(J^{(j)}\right)^{\sigma\lambda} .\end{aligned} \tag{2.54}$$



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### 2.4 The timelike case

As in [11, 13], it is useful to distinguish two cases depending on whether the scalar $f$ vanishes everywhere or not. In the former "null case", the vector $V$ is globally a null Killing vector. We shall not consider this case here — it should be straightforward to analyze using the methods of [11, 13]. In the latter "timelike case" there is some open set $\mathcal{U}$ in which $f$ is non-vanishing and hence $V$ is a timelike Killing vector field. There is no loss of generality in assuming $f > 0$ in $\mathcal{U}$ [11]. We shall analyze the constraints imposed by supersymmetry in the region $\mathcal{U}$.

Introduce coordinates $(t, x^m)$ such that $V = \partial/\partial t$. The metric can then be written locally as

$$ds^2 = f^2(dt + \omega)^2 - f^{-1} h_{mn} dx^m dx^n \,. \tag{2.55}$$

The metric $h_{mn}$ can be regarded as the metric on a four dimensional riemannian manifold, which we shall refer to as the "base space" $B$. $\omega$ is a 1-form on $B$. Since $V$ is Killing, $f$, $\omega$ and $h$ are independent of $t$. We shall reduce the necessary and sufficient conditions for supersymmetry to a set of equations on $B$. Let

$$e^0 = f(dt + \omega) \,. \tag{2.56}$$

We choose the orientation of $B$ so that $e^0 \wedge \eta_4$ is positively oriented in five dimensions, where $\eta_4$ is the volume form of $B$. The two form $d\omega$ can be split into self-dual and anti-self-dual parts on $B$:

$$f d\omega = G^+ + G^- \tag{2.57}$$

where the factor of $f$ is included for convenience.

Equation (2.34) implies that the 2-forms $J^{(i)}$ can be regarded as 2-forms on the base space and equation (2.35) implies that they are anti-self-dual:

$$\star_4 J^{(i)} = -J^{(i)} \,, \tag{2.58}$$

where $\star_4$ denotes the Hodge dual on $B$. Equation (2.36) can be written

$$J^{(i)}{}_m{}^p J^{(j)}{}_p{}^n = -\delta^{ij}\delta_m{}^n + \epsilon_{ijk} J^{(k)}{}_m{}^n \tag{2.59}$$

where indices $m, n, \ldots$ have been raised with $h^{mn}$, the inverse of $h_{mn}$. This equation shows that the $J^{(i)}$'s satisfy the algebra of imaginary unit quaternions, i.e., $B$ admits an almost hyper-Kähler structure, just as in [11, 13].

To proceed, we use (2.39) and (2.41) to obtain

$$\begin{aligned} X_I F^I &= de^0 - \frac{2}{3} G^+ - 2\chi f^{-1} V_I X^I J^{(1)} \\ &= -f^{-1} e^0 \wedge df + \frac{1}{3} G^+ + G^- - 2\chi f^{-1} V_I X^I J^{(1)} \,. \end{aligned} \tag{2.60}$$

From (2.42) we find that

$$\begin{aligned} \nabla_m J^{(1)}_{np} &= 0 \\ \nabla_m J^{(2)}_{np} &= P_m J^{(3)}_{np} \\ \nabla_m J^{(3)}_{np} &= -P_m J^{(2)}_{np} \,, \end{aligned} \tag{2.61}$$



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where $\nabla$ is the Levi-Civita connection on $B$ and we have defined

$$P_m = 3\chi V_I (A^I{}_m - f X^I \omega_m) \,. \tag{2.62}$$

From (2.59) and (2.61) we conclude that, in the gauged theory, the base space is Kähler, with Kähler form $J^{(1)}$. In the ungauged theory, it is hyper-Kähler with Kähler forms $J^{(i)}$. Again, this is all precisely as in the minimal theories [11, 13].

We are primarily interested in the gauged theory so henceforth we shall assume $\chi \neq 0$. Proceeding as in [13], note that we can invert (2.61) to solve for $P$:

$$P_m = \frac{1}{8}\left( J^{(3)np} \nabla_m J^{(2)}_{np} - J^{(2)np} \nabla_m J^{(3)}_{np} \right) , \tag{2.63}$$

from which it follows that

$$dP = \mathcal{R} \,, \tag{2.64}$$

where $\mathcal{R}$ is the Ricci-form of the base space $B$ defined by

$$\mathcal{R}_{mn} = \frac{1}{2} J^{(1)pq} R_{pqmn} \tag{2.65}$$

and $R_{pqmn}$ denotes the Riemann curvature tensor of $B$. Hence, once $B$ has been determined, $P_m$ is determined up to a gradient. An argument in [13] shows that the existence of $J^{(2,3)}$ obeying equations (2.59) and (2.61) is a consequence of $B$ being Kähler, and contains no further information.

Next we examine (2.48). It is convenient to write

$$F^I = -f^{-1} e^0 \wedge d(f X^I) + \Psi^I + \Theta^I + X^I G^+ \tag{2.66}$$

where $\Psi^I$ is an anti-self-dual 2-form on $B$ and $\Theta^I$ is a self-dual 2-form on $B$. Equation (2.60) implies

$$X_I \Theta^I = -\frac{2}{3} G^+ \tag{2.67}$$

and

$$X_I \Psi^I = G^- - 2\chi f^{-1} V_I X^I J^{(1)} \,. \tag{2.68}$$

Now (2.48) determines $\Psi^I$:

$$\Psi^I = X^I G^- - 9\chi f^{-1} C^{IJK} V_J X_K J^{(1)} \,, \tag{2.69}$$

hence

$$F^I = d(X^I e^0) + \Theta^I - 9\chi f^{-1} C^{IJK} V_J X_K J^{(1)} \,. \tag{2.70}$$

$\Theta^I$ is not constrained by the dilatino equation. Finally, from (2.64) together with (2.62) we have the following identity

$$3\chi V_I \Theta^I - 27\chi^2 f^{-1} C^{IJK} V_I V_J X_K J^{(1)} = \mathcal{R} \,. \tag{2.71}$$

Contracting this expression with $J^{(1)}$ we obtain

$$f = -\frac{108\chi^2}{R} C^{IJK} V_I V_J X_K \,, \tag{2.72}$$



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where $R$ is the Ricci scalar of $B$, and hence

$$\mathcal{R} - \frac{1}{4} R J^{(1)} = 3\chi V_I \Theta^I \,. \tag{2.73}$$

Finally, note that equations (2.37) and (2.38) imply that the spinor obeys the projections

$$\gamma^0 \epsilon^a = \epsilon^a \,, \tag{2.74}$$

$$J^{(1)}_{AB} \Gamma^{AB} \epsilon^a = -4\epsilon^{ab}\epsilon^b \,, \tag{2.75}$$

where indices $A, B$ refer to a vierbein $e^A$ on the base space, and $\Gamma^A$ are gamma matrices on the base space given by $\Gamma^A = \pm i\gamma^A$. These projections are not independent: (2.75) implies (2.74).

So far we have been discussing constraints on the spacetime geometry and matter fields that are necessary for the existence of a Killing spinor. We shall now argue that these constraints are also sufficient. Assume that we are given a metric of the form (2.55) for which the base space $B$ is Kähler. Let $J^{(1)}$ denote the Kähler form. Assume that $f$ is given in terms of $X^I$ by equation (2.72) and that the field strengths are given by equation (2.70) where $\Theta^I$ obeys equations (2.67) and (2.73). Now consider a spinor $\epsilon^a$ satisfying the projection (2.75). It is straightforward to show this will automatically satisfy the dilatino equation (2.17). In the basis $(e^0, f^{-1/2}e^A)$, the gravitino equation (2.12) reduces to

$$\partial_t \epsilon^a = 0 \tag{2.76}$$

and

$$\nabla_m \eta^a + \frac{1}{2} P_m \epsilon^{ab} \eta^b = 0 \,, \tag{2.77}$$

where

$$\eta^a = f^{-\frac{1}{2}} \epsilon^a \,. \tag{2.78}$$

The Kähler nature of $B$ guarantees the existence of a solution to equation (2.77) obeying (2.75) without any further algebraic restrictions [22]. Therefore the above conditions on the bosonic fields guarantee the existence of a Killing spinor, i.e., they are both necessary and sufficient for supersymmetry. The only projection required is (2.75), which reduces the number of independent components of the spinor from 8 to 2 so we have at least 1/4 supersymmetry.[7]

We have presented necessary and sufficient conditions for existence of a Killing spinor. However, we are interested in supersymmetric *solutions* so we also need to impose the Bianchi identity $dF^I = 0$ and Maxwell equations (2.15). Substituting the field strengths (2.70) into the Bianchi identities $dF^I = 0$ gives

$$d\Theta^I = 9\chi C^{IJK} V_J \, d\left(f^{-1} X_K\right) \wedge J^{(1)} \,. \tag{2.79}$$

[7] We would also expect the general null solution to be 1/4 supersymmetric, as in the minimal theory [13]. Examples of such solutions were given in [19]. For timelike solutions of the ungauged theory, we expect that the only projection that must be imposed on a Killing spinor is (2.74) so the solutions will be 1/2 supersymmetric, as in the minimal theory [11].



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Note that

$$\star F^I = -f^{-2} \star_4 d\left(fX^I\right) + e^0 \wedge \left(\Theta^I + X^I G^+ - \Psi^I\right), \tag{2.80}$$

so the Maxwell equations (2.15) reduce to

$$\begin{aligned} d \star_4 d\left(f^{-1}X_I\right) &= -\frac{1}{6}C_{IJK}\Theta^J \wedge \Theta^K + 2\chi f^{-1}G^- \wedge J^{(1)} + \\ &\quad + 6\chi^2 f^{-2}\left(Q_{IJ}C^{JMN}V_M V_N + V_I X^J V_J\right)\eta_4 \end{aligned} \tag{2.81}$$

where $\eta_4$ denotes the volume form of $B$.

Finally, the integrability conditions for the existence of a Killing spinor guarantee that the Einstein equation and scalar equations of motion are satisfied as a consequence of the above equations.

In summary, the general timelike supersymmetric solution is determined as follows. First pick a Kähler 4-manifold $B$. Let $J^{(1)}$ denote the Kähler form and $h_{mn}$ the metric on $B$. Equation (2.72) determines $f$ in terms of $X^I$. Next one has to determine $\omega$, $X^I$ and $\Theta^I$ by solving equations (2.67), (2.73), (2.79) and (2.81) on $B$. The metric is then given by (2.55) and the gauge fields by (2.70).

## 3. Black hole solutions

### 3.1 Derivation of the solutions

Following [5], we take the following ansatz for the Kähler base space of a supersymmetric black hole solution:

$$ds_4^2 = d\rho^2 + a^2\left((\sigma_L^1)^2 + (\sigma_L^2)^2\right) + (2aa')^2(\sigma_L^3)^2, \tag{3.1}$$

with Kähler form

$$J^{(1)} = -\epsilon d\left[a^2\sigma_L^3\right], \tag{3.2}$$

where $a = a(\rho)$, $\epsilon = \pm 1$, and $\sigma_L^i$ are right-invariant 1-forms on SU(2). These can be expressed in terms of Euler angles $(\theta, \psi, \phi)$ as

$$\begin{aligned} \sigma_L^1 &= \sin\phi d\theta - \cos\phi\sin\theta d\psi \\ \sigma_L^2 &= \cos\phi d\theta + \sin\phi\sin\theta d\psi \\ \sigma_L^3 &= d\phi + \cos\theta d\psi, \end{aligned} \tag{3.3}$$

where SU(2) is parametrized by taking $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 4\pi$ and $0 \leq \psi \leq 2\pi$. The right-invariant 1-forms obey

$$d\sigma_L^i = -\frac{1}{2}\epsilon_{ijk}\sigma_L^j \wedge \sigma_L^k. \tag{3.4}$$

The surfaces of constant $\rho$ are homogeneous, with a transitively acting $\mathrm{U}(1)_L \times \mathrm{SU}(2)_R$ isometry group, The $\mathrm{U}(1)_L$ generated by $\partial/\partial\phi$ is manifestly a symmetry and the $\mathrm{SU}(2)_R$ is a symmetry because $\sigma_L^i$ is invariant under the right action of SU(2).

We shall assume $a, a' > 0$ and introduce an orthonormal basis

$$e^1 = d\rho, \qquad e^2 = a\,\sigma_L^1, \qquad e^3 = a\,\sigma_L^2, \qquad e^4 = 2aa'\,\sigma_L^3 \tag{3.5}$$

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with volume form $\eta_4 = e^1 \wedge e^2 \wedge e^3 \wedge e^4$. We then have

$$J^{(1)} = -\epsilon(e^1 \wedge e^4 - e^2 \wedge e^3)\,, \tag{3.6}$$

which is obviously anti-self-dual.

As in [5], we adopt the Ansatz

$$\omega = w(\rho)\sigma^3_L\,, \tag{3.7}$$

which gives

$$G^+ = \frac{fa}{4a'}\partial_\rho(a^{-2}w)(e^1 \wedge e^4 + e^2 \wedge e^3) \tag{3.8}$$

and

$$G^- = \frac{f}{4a^3a'}\partial_\rho(a^2w)(e^1 \wedge e^4 - e^2 \wedge e^3)\,. \tag{3.9}$$

We also take

$$X^I = X^I(\rho) \tag{3.10}$$

and

$$A^I = X^I e^0 + U^I(\rho)\sigma^3_L\,, \tag{3.11}$$

for some functions $U^I$. $f$ is determined by (2.72):

$$f = \frac{54\chi^2a^2a'C^{IJK}V_IV_JX_K}{a^2a''' - a' + 7aa'a'' + 4(a')^3}\,. \tag{3.12}$$

Note that

$$F^I = d(X^Ie^0) + \frac{\partial_\rho U^I}{2aa'}\,e^1 \wedge e^4 - \frac{U^I}{a^2}\,e^2 \wedge e^3\,. \tag{3.13}$$

Comparing (3.13) with (2.70) we find

$$\Theta^I = \frac{a}{4a'}\partial_\rho(a^{-2}U^I)(e^1 \wedge e^4 + e^2 \wedge e^3) \tag{3.14}$$

and

$$\partial_\rho(a^2U^I) = 36\epsilon\chi f^{-1}a^3a'C^{IJK}V_JX_K\,. \tag{3.15}$$

This equation is sufficient to ensure that the Bianchi identity (2.79) holds automatically. Equation (2.67) gives

$$f^{-1}X_I\partial_\rho(a^{-2}U^I) = -\frac{2}{3}\partial_\rho(a^{-2}w) \tag{3.16}$$

and from (2.73) we find

$$3\chi V_IU^I = \epsilon(-1 + 2aa'' + 4(a')^2)\,. \tag{3.17}$$

Lastly, we compute the Maxwell equations (2.81); we obtain

$$\partial_\rho\left[a^3a'\partial_\rho(f^{-1}X_I) + \epsilon\chi a^2wV_I + \frac{1}{12}C_{IJK}U^JU^K\right] = 0\,. \tag{3.18}$$

To find a solution to these equations we make the guess

$$f^{-1}X_I = \bar{X}_I + \frac{q_I}{4a^2}\,, \tag{3.19}$$



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where $\bar{X}_I$ are the constant values of the scalars in the maximally supersymmetric $AdS_5$ solution and $q_I$ are constants. Since

$$C^{IJK} X_I X_J X_K = \frac{2}{9} \tag{3.20}$$

we must have

$$f^{-3} = \frac{9}{2} C^{IJK} \left(\bar{X}_I + \frac{q_I}{4a^2}\right) \left(\bar{X}_J + \frac{q_J}{4a^2}\right) \left(\bar{X}_K + \frac{q_K}{4a^2}\right), \tag{3.21}$$

which implies

$$f = \left(1 + \frac{\alpha_1}{4a^2} + \frac{\alpha_2}{16a^4} + \frac{\alpha_3}{64a^6}\right)^{-1/3}, \tag{3.22}$$

where we have defined the constants

$$\alpha_1 = \frac{27}{2} C^{IJK} \bar{X}_I \bar{X}_J q_K\,, \qquad \alpha_2 = \frac{27}{2} C^{IJK} \bar{X}_I q_J q_K\,, \qquad \alpha_3 = \frac{9}{2} C^{IJK} q_I q_J q_K\,. \tag{3.23}$$

Equation (3.15) then determines

$$U^I = \frac{9\epsilon}{\ell} C^{IJK} \bar{X}_J \left(a^2 \bar{X}_K + \frac{q_K}{2}\right). \tag{3.24}$$

A possible term of the form (constant of integration) times $a^{-2}$ has been set to zero because such a term is not present in the supersymmetric black hole solutions of the minimal theory. Now we can determine $w$ from (3.16):

$$w = \frac{\epsilon}{\ell} \left(w_0 a^2 - \frac{\alpha_1}{2} - \frac{\alpha_2}{16a^2}\right), \tag{3.25}$$

where $w_0$ is a constant of integration. Equation (3.17) can be integrated to give

$$(a')^2 = \frac{a^2}{\ell^2} + \frac{1}{4}\left(1 + \frac{\alpha_1}{\ell^2}\right) + \frac{\kappa}{a^4}\,, \tag{3.26}$$

where $\kappa$ is a constant of integration. Comparison with the minimal theory shows that we must take $\kappa = 0$. We can then integrate to obtain

$$a = \frac{\ell}{2}\sqrt{1 + \frac{\alpha_1}{\ell^2}} \sinh\left(\frac{\rho}{\ell}\right). \tag{3.27}$$

This determines the geometry of the base space: it is the same singular deformation of the Bergmann manifold that appears in the minimal theory [5]. Finally, we require equation (3.18) to be satisfied. Upon using equation (2.2), this reduces to

$$w_0 = -2\,. \tag{3.28}$$

Hence

$$w = -\frac{\epsilon}{\ell}\left(2a^2 + \frac{\alpha_1}{2} + \frac{\alpha_2}{16a^2}\right). \tag{3.29}$$



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### 3.2 Properties of the solution

It is convenient to work with a new radial coordinate $R$ defined by[8]

$$a = \frac{R}{2}. \tag{3.30}$$

The metric is

$$ds^2 = f^2 dt^2 + 2f^2 w\, dt\, \sigma_L^3 - f^{-1} g^{-1} dR^2 - \frac{R^2}{4}\left[f^{-1}((\sigma_L^1)^2 + (\sigma_L^2)^2) + f^2 h(\sigma_L^3)^2\right], \tag{3.31}$$

where $\sigma_L^i$ was defined in equation (3.3) and

$$f = \left(1 + \frac{\alpha_1}{R^2} + \frac{\alpha_2}{R^4} + \frac{\alpha_3}{R^6}\right)^{-1/3}, \tag{3.32}$$

$$w = -\frac{\epsilon R^2}{2\ell}\left(1 + \frac{\alpha_1}{R^2} + \frac{\alpha_2}{2R^4}\right), \tag{3.33}$$

$$g = 1 + \frac{\alpha_1}{\ell^2} + \frac{R^2}{\ell^2}, \tag{3.34}$$

$$h \equiv f^{-3} g - \frac{4}{R^2} w^2 = 1 + \frac{\alpha_1}{R^2} + \frac{1}{R^4}\left(\alpha_2 + \frac{\alpha_3}{\ell^2}\right) + \frac{1}{R^6}\left[\alpha_3\left(1 + \frac{\alpha_1}{\ell^2}\right) - \frac{\alpha_2^2}{4\ell^2}\right], \tag{3.35}$$

where $\epsilon = \pm 1$ determines the sense of rotation. The scalars are

$$X_I = f\left(\bar{X}_I + \frac{q_I}{R^2}\right). \tag{3.36}$$

The gauge fields are

$$A^I = f X^I dt + \left(U^I + f w X^I\right)\sigma_L^3, \tag{3.37}$$

where

$$U^I = \frac{9\epsilon}{4\ell} C^{IJK} \bar{X}_J \left(\bar{X}_K R^2 + 2q_K\right). \tag{3.38}$$

The solution is parametrized by the constants $q_I$, which determine $\alpha_i$ via equation (3.23). Recall that the constants $\bar{X}_I$ are the values of the scalars in the $AdS_5$ vacuum solution, which has radius $\ell$. The solution has isometry group $R \times \mathrm{U}(1)_L \times \mathrm{SU}(2)_R$ where $R$ is generated by the supersymmetric Killing vector field $V = \partial/\partial t$, $\mathrm{U}(1)_L$ by $\partial/\partial\phi$ and the $\mathrm{SU}(2)_R$ arises because $\sigma_L^i$ is invariant under the right action of SU(2). The supersymmetric black hole solutions of the minimal theory have [5] $\alpha_1 = 3R_0^2$, $\alpha_2 = 3R_0^4$ and $\alpha_3 = R_0^6$.

To see that the solution is asymptotically AdS, let

$$\phi' = \phi + \frac{2\epsilon t}{\ell}. \tag{3.39}$$

We then have

$$ds^2 = f^{-1} g h^{-1} dt^2 - f^{-1} g^{-1} dR^2 - \frac{R^2}{4}\left[f^{-1}\left((\sigma_L^{1'})^2 + (\sigma_L^{2'})^2\right) + f^2 h(\sigma_L^{3'} - \Omega dt)^2\right], \tag{3.40}$$

[8]Note that this differs from the coordinate $R$ used in [5] by a factor of $f$.



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where $\sigma_L^{i'}$ is defined in the same way as $\sigma_L^i$ (equation (3.3)) but with $\phi$ replaced by $\phi'$ and

$$\Omega = \frac{2\epsilon}{\ell h}\left\{\frac{1}{R^4}\left(\frac{\alpha_2}{2}+\frac{\alpha_3}{\ell^2}\right)+\frac{1}{R^6}\left[\alpha_3\left(1+\frac{\alpha_1}{\ell^2}\right)-\frac{\alpha_2^2}{4\ell^2}\right]\right\}. \tag{3.41}$$

Now as $R \to \infty$, we have $f, h \to 1$, $\Omega \to 0$ and $g \propto R^2/\ell^2$ so the solution is manifestly asymptotic to $AdS_5$. In these coordinates, the supersymmetric Killing vector field is $V = \partial/\partial t + (2\epsilon/\ell)\partial/\partial\phi'$. The gauge fields have the asymptotic behaviour

$$\begin{aligned} A^I = &\left\{\bar{X}^I - \frac{1}{R^2}\left[\left(\alpha_1+\frac{\alpha_2}{2\ell^2}\right)\bar{X}^I - 9C^{IJK}\bar{X}_J q_K - \frac{9}{2\ell^2}C^{IJK}q_J q_K\right]+\right.\\ &\left.+\mathcal{O}\left(\frac{1}{R^4}\right)\right\}dt + \mathcal{O}\left(\frac{1}{R^2}\right)\sigma_L^{3'}. \end{aligned} \tag{3.42}$$

Note that if $\alpha_2 = \alpha_3 = 0$ then the solution reduces to one of the static, spherically symmetric, nakedly singular, solutions investigated in [17].

To investigate which solutions have regular horizons, we shall attempt to introduce gaussian null coordinates as follows:

$$dt = du - fg^{-1}h\,dr\,, \qquad d\phi = d\phi'' - \frac{4}{R^2}fg^{-1}w\,dr\,, \qquad dR = f\sqrt{h}\,dr\,. \tag{3.43}$$

The line element becomes

$$ds^2 = f^2du^2 - 2dudr + 2f^2w\,du\,\sigma_L^{3''} - \frac{R^2}{4}\left[f^{-1}\left((\sigma_L^{1''})^2+(\sigma_L^{2''})^2\right)+f^2h(\sigma_L^{3''})^2\right], \tag{3.44}$$

where $\sigma_L^{i''}$ is defined in the same way as $\sigma_L^i$ (equation (3.3)) but with $\phi$ replaced by $\phi''$. The supersymmetric Killing vector field is $V = \partial/\partial u$. In order for there to be a regular horizon at $R = 0$ we need $R^2 f^{-1}$ to approach a positive constant as $R \to 0$. This requires

$$\alpha_3 > 0\,. \tag{3.45}$$

We also need $R^2 f^2 h$ to approach a positive constant, which requires

$$\alpha_3\left(1+\frac{\alpha_1}{\ell^2}\right)-\frac{\alpha_2^2}{4\ell^2} > 0\,. \tag{3.46}$$

We then find that $r \propto R^2$ as $R \to 0$ and that $f$ and $f^2 w$ are $\mathcal{O}(r)$ as $r \to 0$, which guarantees a regular horizon. Hence, subject to the above restrictions, our solution has a regular horizon at $R = 0$. In order to avoid problems in $R > 0$ we must also demand that $f$, $g$ and $h$ be positive for $R > 0$, which imposes further restrictions on $\alpha_i$.

Spatial cross-sections of the event horizon have the geometry of a squashed $S^3$ with area

$$\mathcal{A} = 2\pi^2\sqrt{\alpha_3\left(1+\frac{\alpha_1}{\ell^2}\right)-\frac{\alpha_2^2}{4\ell^2}}\,. \tag{3.47}$$

The angular velocity of the event horizon with respect to the stationary frame at infinity can be calculated as in [5], giving

$$\Omega_H = \frac{2\epsilon}{\ell}\,. \tag{3.48}$$



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We note that if we take the limit $\ell \to \infty$ with $q_I$ held fixed then our solutions reduce to *static* supersymmetric black hole solutions of the ungauged supergravity theory, which were first obtained in [23].

We can calculate the mass and angular momentum of our solutions using the definitions of Ashtekar and Das (AD) [24]. The AD mass is associated with the symmetry of the conformal boundary generated by $\partial/\partial t$ in the coordinates (3.40):

$$M = \frac{\pi}{4G}\left(\alpha_1 + \frac{3\alpha_2}{2\ell^2} + \frac{2\alpha_3}{\ell^4}\right), \tag{3.49}$$

and the AD angular momentum is associated with $-\partial/\partial\phi'$:

$$J = \frac{\epsilon\pi}{8G\ell}\left(\alpha_2 + \frac{2\alpha_3}{\ell^2}\right). \tag{3.50}$$

Just as in the minimal theory, this really corresponds to equal angular momenta $J_1 = J_2 = J$ in two orthogonal 2-planes.

We have used the AD approach to define mass and angular momentum. However, it has been argued [25, 26] that the conserved quantities of the AD approach do not correctly reproduce the (anomalous) transformation law of the CFT energy-momentum tensor [27], so the AD mass $M$ should not be interpreted as dual to CFT energy. Instead one should use an alternative approach based on the "holographic stress tensor" (HST) [28, 29], which does transform correctly.

If $M$ cannot be interpreted as dual to CFT energy then what is its CFT interpretation? To answer this, we first note that the AD definitions only apply to spacetimes that are asymptotically AdS whereas the HST approach applies more generally to spacetimes that are merely asymptotically locally AdS with a well-defined conformal boundary. Therefore, when $M$ is defined, the dual CFT must live on $R \times S^3$ whereas the HST energy $E$ can be defined for many different CFT background geometries. Now $R \times S^3$ is precisely the background for which the CFT state/operator correspondence applies [6]. Under this correspondence, the energy of a state is equal to the dimension of the corresponding operator plus an anomalous term that can be interpreted as the Casimir energy on $R \times S^3$. This suggests that we should identify the AD mass of a bulk solution with the dimension of the corresponding local CFT operator(s):

$$\Delta = M\ell. \tag{3.51}$$

Of course, this should only be regarded as the leading term in a large $N$ expansion. More precisely, we mean that $\Delta/N^2$ and $M\ell/N^2$ tend to the same limit as $N \to \infty$.

In the few examples for which $M$ and $E$ have both been calculated, it has been found that they differ precisely by the Casimir energy of the CFT on $R \times S^3$, which is evidence in favour of the above interpretation.[9] It would be interesting to see whether this could be

[9]It was claimed in [30] that the difference is more complicated for the rotating black hole solutions of [31]. However, the HST results of [30] (following the earlier [32]) correspond to the stress tensor of a CFT in $R \times S^3$ with a non-product metric [32], for which one would expect the Casimir energy to be more complicated anyway.



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proved more generally. It would also be interesting to calculate the HST for our solutions. This was done for solutions of the minimal theory in [5], where it was found that $M$ and $E$ differ by the Casimir energy and that the two approaches give the same value for $J$. In the theory under consideration here, the counterterms required for calculating the HST do not appear to have been derived yet.

We shall define conserved electric charges by:

$$Q_I = \frac{1}{8\pi G}\int Q_{IJ} \star F^J \,, \tag{3.52}$$

where the integral is taken over a spatial three sphere at infinity. Calculating this on a surface of constant $t$ in the asymptotically AdS coordinates gives

$$Q_I = \frac{\pi}{G}\left(\frac{3}{4}q_I - \frac{3\alpha_2}{8\ell^2}\bar{X}_I + \frac{9}{8\ell^2}C_{IJK}\bar{X}^J C^{KLM} q_L q_M\right). \tag{3.53}$$

This implies

$$\bar{X}^I Q_I = \frac{\pi}{4G}\left(\alpha_1 + \frac{\alpha_2}{2\ell^2}\right), \tag{3.54}$$

so we have the BPS equality

$$M - \frac{2}{\ell}|J| = |\bar{X}^I Q_I|\,. \tag{3.55}$$

It would be interesting to look for non-extremal generalizations of these solutions. In general, such solutions will carry two independent angular momenta, which will make them rather complicated (see [31] for uncharged solutions). However, the solutions should simplify when the angular momenta are equal, with the isometry group being enhanced from $R\times \mathrm{U}(1)^2$ to $R\times \mathrm{U}(1)_L \times \mathrm{SU}(2)_R$ (as for the supersymmetric solutions). If the metric is written using right-invariant forms on SU(2) then the metric components will all be functions of a single radial coordinate, so finding these solutions should not be difficult.

### 3.3 Solutions of the $\mathrm{U}(1)^3$ theory

We are primarily interested in solutions that can be oxidized to yield asymptotically $AdS_5\times S^5$ solutions of type-IIB supergravity. We therefore consider the theory with $\mathrm{U}(1)^3$ gauge group obtained by taking indices $I, J, K$ to run from 1 to 3 and with $C_{IJK} = 1$ if $(IJK)$ is a permutation of (123) and $C_{IJK} = 0$ otherwise. The constraint on the scalars is then

$$X^1 X^2 X^3 = 1 \tag{3.56}$$

and we have

$$Q_{IJ} = \frac{9}{2}\mathrm{diag}\left((X_1)^2, (X_2)^2, (X_3)^2\right). \tag{3.57}$$

We also take

$$V_I = \frac{\xi}{3}\,. \tag{3.58}$$

Solutions of this theory can be oxidized to solutions of type-IIB supergravity as described in [8].[10] In this theory,

$$\bar{X}_I = \frac{1}{3}\,, \qquad \bar{X}^I = 1\,. \tag{3.59}$$

[10] Our gauge fields have the same normalization as those of [8]. Denote the scalars of [8] by $\tilde{X}_I$. They are related to our scalars by $\tilde{X}_I = 1/(3X_I)$. The coupling of [8] is related to our constants by $g = \chi\xi = 1/\ell$.



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It is convenient to define rescaled parameters for the black hole solutions:

$$q_I = \frac{\mu_I}{3} \tag{3.60}$$

so that

$$\alpha_1 = \mu_1 + \mu_2 + \mu_3\,, \qquad \alpha_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1\,, \qquad \alpha_3 = \mu_1\mu_2\mu_3\,. \tag{3.61}$$

We then have

$$f = (H_1 H_2 H_3)^{-1/3}\,, \tag{3.62}$$

where

$$H_I = 1 + \frac{\mu_I}{R^2}\,. \tag{3.63}$$

The scalars are given by

$$X_I = \frac{1}{3} H_I \left(H_1 H_2 H_3\right)^{-1/3}\,. \tag{3.64}$$

We need $\mu_I > 0$ to guarantee $\alpha_3 > 0$ and $f > 0$ for $R > 0$. The only remaining restriction on $\mu_I$ for the solution to describe a black hole is equation (3.46), which can be written

$$4\mu_1\mu_2\mu_3\left(\mu_1 + \mu_2 + \mu_3 + \ell^2\right) > \left(\mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1\right)^2\,. \tag{3.65}$$

This constraint is non-trivial: e.g. it is not satisfied if we take $\mu_1 = \mu_2 \gg \mu_3$. The mass, angular momentum and charges can be obtained from the expressions above. The charges simplify to

$$\begin{aligned}
Q_1 &= \frac{\pi}{4G}\left[\mu_1 + \frac{1}{2\ell^2}\left(\mu_1\mu_2 + \mu_1\mu_3 - \mu_2\mu_3\right)\right] \\
Q_2 &= \frac{\pi}{4G}\left[\mu_2 + \frac{1}{2\ell^2}\left(\mu_2\mu_3 + \mu_2\mu_1 - \mu_3\mu_1\right)\right] \\
Q_3 &= \frac{\pi}{4G}\left[\mu_3 + \frac{1}{2\ell^2}\left(\mu_3\mu_1 + \mu_3\mu_2 - \mu_1\mu_2\right)\right].
\end{aligned} \tag{3.66}$$

It is possible to set one, but not two, of the charges to zero consistently with the constraint (3.65).

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